ISRAEL JOURNAL OF MATHEMATICS 145 (2005), 375–379

A CORRIGENDUM TO THE ARTICLE: RAMSEY-MILMAN PHENOMENON, URYSOHN METRIC SPACES, AND EXTREMELY AMENABLE GROUPS

BY

VLADIMIR PESTOV

Department of Mathematics and Statistics, University of Ottawa
585 King Edward Avenue, Ottawa, Ontario, Canada K1N6N5
e-mail: vpest283@uottawa.ca
URL: http://aix1.uottawa.ca/~vpest283

As kindly pointed out to me by C. Ward Henson and Slawomir Solecki, the proof of one of the main technical results (Theorem 3.2) in the paper [3] is flawed. To quote from Prof. Henson's message: "Suppose a, b are elements of $F = F^{(3)}_{m+n}$ that have the minimum distance δ from each other in the ρ' metric, and let w be any word in F. Since the metric d is bi-invariant, the conjugate $v = wab^{-1}w^{-1}$ of ab^{-1} has d-distance δ from the identity. But it seems clear that the reduced length of v could be made arbitrarily large by choosing w correctly. This contradicts what you claim in (8)."

Fortunately, the result is not particularly deep, and here is a corrected proof of the statement.

As in [3], we say that a metric space X is **indexed** by a set I if there is a surjection $f_X \colon I \to X$. We will call the pair (X, f_X) an **indexed metric space**. Two metric spaces, X and Y, indexed with the same set I are **ε-isometric** if for every $i, j \in I$ the distances $d_X(f_X(i), f_X(j))$ and $d_Y(f_Y(i), f_Y(j))$ differ by at most ε.

Here is the result in question.

THEOREM 3.2: *Let $g_1, \ldots, g_m$ be a finite family of isometries of a metric space X. Then for every $\varepsilon > 0$ and every finite collection $x_1, \ldots, x_n$ of elements of X there exist a finite metric space $\widetilde{X}$, elements $\tilde{x}_1, \ldots, \tilde{x}_n$ of $\tilde{X}$, and isometries $\tilde{g}_1, \ldots, \tilde{g}_m$ of $\tilde{X}$ such that the indexed metric spaces*

$$\{g_j \cdot x_i \colon i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$$

Received August 11, 2003

and

$$\{\tilde{g}_j \cdot \tilde{x}_i\colon i = 1,2,\ldots,n, j = 1,2,\ldots,m\}$$

are ε-isometric.

Proof: Without loss in generality, we can assume that X is separable (in fact, even countable). Such an X can be g-embedded into the Urysohn space $\mathbb{U}$ (see [4], or else Prop. 4.1 in [3]), and therefore we can further assume that $X = \mathbb{U}$.

Choose any element $\xi \in \mathbb{U}$ and isometries $g_{m+1}, g_{m+2}, \ldots, g_{m+n}$ of $\mathbb{U}$ with the property $g_{m+i}(\xi) = x_i$, $i = 1,2,\ldots,n$.

Denote by F_{m+n} the free non-abelian group on generators

$$g_1, \ldots, g_m, g_{m+1}, \ldots, g_{m+n}.$$

The group F_{m+n} acts on $\mathbb{U}$ by isometries.

The formula

$$d(g,h) := d_{\mathbb{U}}(g(\xi), h(\xi)), \quad g,h \in F_{m+n},$$

where $d_{\mathbb{U}}$ denotes the metric on the Urysohn space, defines a left-invariant pseudometric d on the group F_{m+n}:

$$\begin{aligned} d(xg,xh) &= d_{\mathbb{U}}(xg(\xi), xh(\xi)) \\ &= d_{\mathbb{U}}(x(g(\xi)), x(h(\xi))) \\ &= d_{\mathbb{U}}(g(\xi), h(\xi)) \\ &= d(g,h). \end{aligned}$$

The indexed metric subspace $\{g_j \cdot x_i\colon i = 1,2,\ldots,n, j = 1,2,\ldots,m\}$ of $\mathbb{U}$ is isometric to the metric subspace $\{g_j \cdot g_{m+i}\colon i = 1,2,\ldots,n, j = 1,2,\ldots,m\}$ of (F_{m+n}, d). Indeed,

$$\begin{aligned} d_{\mathbb{U}}(g_j \cdot x_i, g_k \cdot x_l) &= d_{\mathbb{U}}(g_j \cdot g_{m+i}(\xi), g_k \cdot g_{m+l}(\xi)) \\ &= d(g_j g_{m+j}, g_k g_{m+i}). \end{aligned}$$

Notice also that the latter subspace is contained in the set $F^{(2)}_{m+n}$ of all words having reduced length ≤ 2. (The reduced length will always mean that with regard to the generators $g_1, \ldots, g_m, g_{m+1}, \ldots, g_{m+n}$.)

By adding to d a left-invariant metric on F_{m+n} taking sufficiently small values on pairs of elements of $F^{(2)}_{m+n}$, we can assume without loss in generality that d is a left-invariant metric on F_{m+n}. (For instance, add a metric whose only non-zero value is $\varepsilon/3$.)

Form the Cayley graph Γ of F_{m+n} with regard to the set of generators $Y = F^{(4)}_{m+n}$. Vertices of Γ are elements of F_{m+n}, and $x, y \in F_{m+n}$ are adjacent if and only if $x^{-1}y \in F^{(4)}_{m+n}$. This graph is connected. Now make Γ into a weighted graph by assigning to an edge (a, b), $a^{-1}b \in F^{(4)}_{m+n}$ the value $d(a, b) \equiv d(a^{-1}b, e)$.

Denote by ρ the path metric of the weighted graph Γ. Its value for $x, y \in F_{m+n}$ is given by

$$\rho(x,y) = \inf \sum_{i=0}^{N-1} d(a_i, a_{i+1}), \tag{1}$$

where the infimum is taken over all natural N and all finite sequences $x = a_0, a_1, \ldots, a_{N-1}, a_N = y$, with the property $a_i^{-1}a_{i+1} \in F^{(4)}_{m+n}$ for all i.

It is easily seen that ρ is a left-invariant metric on the group F_{m+n}.

Generally, $\rho \geq d$, but restrictions of ρ and d to $F^{(2)}_{m+n}$ coincide.

If one denotes by $\delta > 0$ the minimal value of $d(a, b)$ as $a, b \in F^{(2)}_{m+n}$ and $a \neq b$, then $\rho(x, y) \geq \delta d_w(x, y)$, where d_w denotes the word metric with respect to the set of generators $Y = F^{(4)}_{m+n}$. In particular, if an $x \in F_{m+n}$ has reduced length $l = l(x)$, then $d_w(x) \geq l/4$ and accordingly $\rho(x, e) \geq \delta l/4$. (As a consequence, the infimum in Eq. (1) is always achieved.)

Let Δ denote the maximal value of the metric ρ between pairs of elements of $F^{(2)}_{m+n}$. Choose a natural number N so large that $\delta(N-4)/4 \geq \Delta$; for instance, set $N = 4\lfloor \Delta/\delta \rfloor + 4$.

Every free group is residually finite, that is, admits a separating family of homomorphisms into finite groups. (Cf. e.g. [1], Ch. 7, exercise 5.) Using this fact, choose a normal subgroup $H \triangleleft F_{m+n}$ of finite index so that $H \cap F^{(N)}_{m+n} = \{e\}$.

The formula

$$\begin{aligned} \bar\rho(xH, yH) &:= \inf_{h_1,h_2\in H} \rho(xh_1, yh_2) \\ &\equiv \inf_{h_1,h_2\in H} \rho(h_1x, h_2y) \\ &\equiv \inf_{h\in H} \rho(hx, y) \end{aligned} \tag{2}$$

defines a left-invariant pseudometric on the finite factor-group F_{m+n}/H. The triangle inequality follows from the fact that, for all $h' \in H$,

$$\begin{aligned} \bar\rho(xH, yH) &= \inf_{h\in H} \rho(hx, y) \\ &\leq \inf_{h\in H} [\rho(hx, h'z) + \rho(h'z, y)] \\ &= \inf_{h\in H} \rho(hx, h'z) + \rho(h'z, y) \end{aligned}$$

$$= \inf_{h \in H} \rho(h'^{-1}hx, z) + \rho(h'z, y)$$
$$= \bar{\rho}(xH, zH) + \rho(h'z, y),$$

and the infimum of the r.h.s. taken over all $h' \in H$ equals $\bar{\rho}(xH,zH)+\bar{\rho}(zH,yH)$. Left-invariance of $\bar{\rho}$ is obvious.

Let $x, y \in F^{(2)}_{m+n}$. Closely approximate the infimum in Eq. (2) by some value $\rho(xh_1, yh_2)$ with $h_1, h_2 \in H$; then

$$\rho(xh_1, yh_2) = \rho(y^{-1}xh_1x^{-1}y \cdot y^{-1}x, h_2) = \rho(y^{-1}x, h_3),$$

where $h_3 = y^{-1}xh_1^{-1}x^{-1}yh_2 \in H$.

The value $\rho(y^{-1}x, h_3)$, $h_3 \in H$, cannot get smaller than $d(y^{-1}x, e) = d(x, y)$. Indeed, unless $h_3 = e$ (in which case $\rho(y^{-1}x, h_3) = \rho(x, y) = d(x, y)$), one has $l(h_3) \geq N$ and so the word distance from $y^{-1}x$ to h_3 is at least $N - 4$, and $\rho(y^{-1}x, h) \geq \delta(N-4)/4 \geq \Delta \geq d(x, y)$.

We conclude: the restriction of the factor-homomorphism

$$\pi: F_{m+n} \ni x \mapsto xH \in F_{m+n}/H$$

to $F^{(2)}_{m+n}$ is an isometry.

One can now perturb the pseudometric on F_{m+n}/H by adding to it a left-invariant metric taking very small values (e.g. taking the only non-zero value $\varepsilon/3$) so as to replace $\bar{\rho}$ with a left-invariant metric, $\tilde{\rho}$.

Take now $\tilde{X} = (F_{m+n}/H, \tilde{\rho})$, $\tilde{x}_i = \pi(g_{m+i}) \in \tilde{X}$, $i = 1, 2, \ldots, n$, and let $\tilde{g}_j$ be left translates made by the elements $\pi(g_j)$, $j = 1, 2, \ldots, m$, in the finite group F_{m+n}/H. The indexed metric space

$$\{g_j \cdot g_{m+i}: i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$$

of (F_{m+n}, d) is ε-isometric to the metric subspace

$$\{\pi(g_j) \cdot \pi(g_{m+i}): i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$$

of $(F_{m+n}/H, \tilde{\rho})$. Consequently, the conclusion of the Theorem is verified. ∎

Remark: Prof. Henson has also pointed out to me that in the particular case of path metric spaces associated to a graph the above result (Theorem 3.2) follows from earlier results by Hrushovski [2].

ACKNOWLEDGEMENT: The author is most grateful to Professors Henson and Solecki for their comments, as well as to all other participants in the seminar series "Polish group actions and extremely amenable groups" organized by Prof. Solecki in Spring 2003 at the University of Illinois.

References

[1] M. Hall, jr., *The Theory of Groups*, AMS Chelsea Publ., American Mathematical Society, Providence, RI, 1999 reprint of the 1959 original.

[2] E. Hrushovski, *Extending partial isomorphisms of graphs*, Combinatorica **12** (1992), 411–416.

[3] V. Pestov, *Ramsey–Milman phenomenon, Urysohn metric spaces, and extremely amenable groups*, Israel Journal of Mathematics **127** (2002), 317–358.

[4] V. V. Uspenskij, *On subgroups of minimal topological groups*, Ohio University 1998 preprint, e-print available at `http://arXiv.org/abs/math.GN/0004119`.